Caringbah High School

2016

Trial HSC Examination

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen (Black pen is preferred)
- Board-approved calculators may be used
- A Board-approved reference sheet is provided for this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2 – 5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6 – 14

90 marks

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Question 1 - 10 (1 mark each) Answer on the page provided.

1 If $a = 1 - 2c$, which expression has c been correctly made the subject?

(A) $c = \dfrac{-1-a}{2}$

(B) $c = \dfrac{1-a}{2}$

(C) $c = \dfrac{a-1}{2}$

(D) $c = \dfrac{a+1}{2}$

2 If $f(x) = 2x^2 - 3x + 4$, what is the value of $f(1) - f(-1)$?

(A) -6

(B) -2

(C) 6

(D) 2

3 The number of worms W in a worm farm, at time t is given by $W = 3000e^{-kt}$,

where k is a positive constant.

Over time, which expression describes the change in the number of worms?

(A) decreasing at a constant rate

(B) increasing at a constant rate

(C) decreasing exponentially

(D) increasing exponentially

4 The second term and the fifth term of a geometric sequence are -3 and 192 respectively. What is the common ratio of the sequence?

(A) 4

(B) -4

(C) 8

(D) -8

5 Which of the following circles has the lines $x = 1$, $x = 5$, $y = 3$, and $y = 7$ as its tangents?

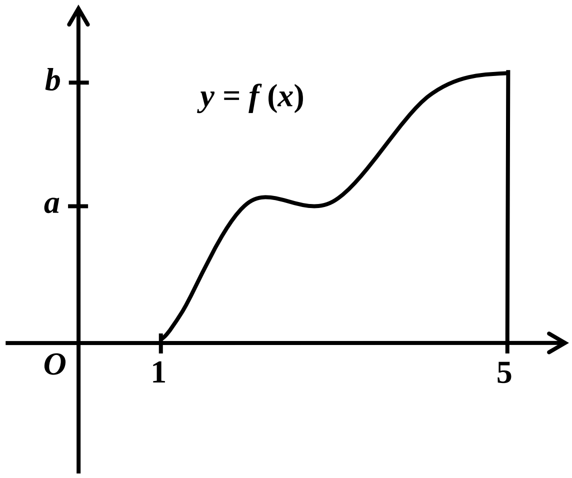
(A) $(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = 4$

(B) $\left(x - \frac{5}{2}\right)^2 + (y-5)^2 = 4$

(C) $(x-5)^2 + (y-3)^2 = 4$

(D) $(x-3)^2 + (y-5)^2 = 4$

6



Using Simpson's rule with 3 function values, which expression best represents the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = 1$ and $x = 5$?

(A) $\frac{2}{3}(1 + 4a + b)$

(B) $\frac{1}{2}(1 + 4a + b)$

(C) $\frac{1}{2}(4a + b)$

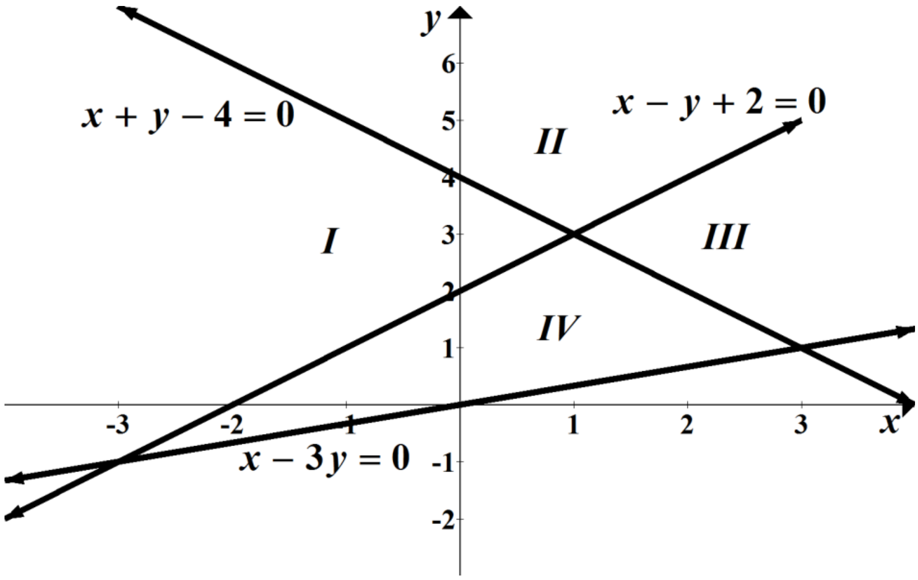
(D) $\frac{2}{3}(4a + b)$

7 How many points of intersection of the graph of $y = \sin x$ and $y = 1 + \cos x$ lie between 0 and 2π?

(A) 1 (B) 2

(C) 3 (D) 4

8



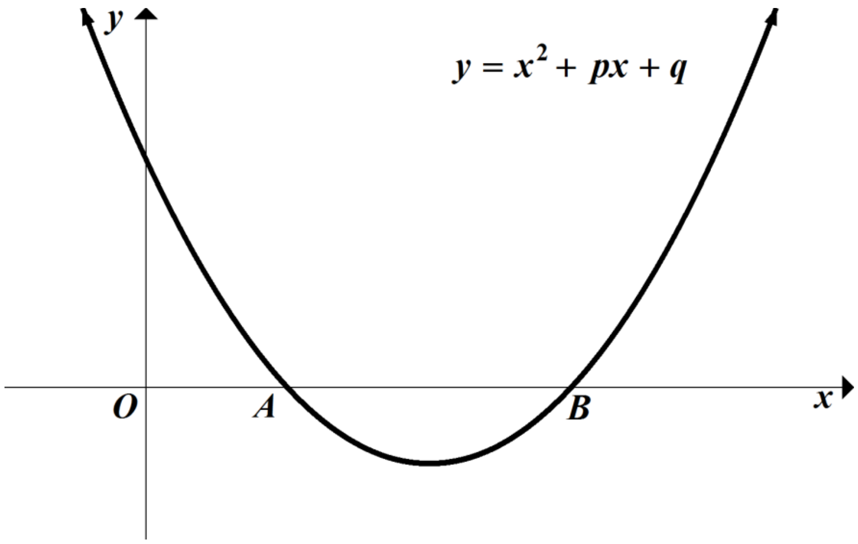
In the figure above, which region represents the solution to the following inequalities?

$$\left.\begin{array}{r} x - 3y < 0 \\ x - y + 2 > 0 \\ x + y - 4 > 0 \end{array}\right\}$$

(A) *I* (B) *II*

(C) *III* (D) *IV*

9



In the figure above, the graph of $y = x^2 + px + q$ cuts the x-axis at A and B.

Which is the value of $OA + OB$?

(A) $-p$

(B) p

(C) $-q$

(D) q

10 If $\log_{10} x, \log_{10} y, \log_{10} z$ form an arithmetic progression, which one of the following relationships hold true?

(A) $y = 10^{\frac{x+z}{2}}$

(B) $y = \frac{x+z}{2}$

(C) $y^2 = x + z$

(D) $y^2 = xz$

END OF MULTIPLE CHOICE QUESTIONS

Section II

90 marks
Attempt all questions 11−16
Allow about 2 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11−16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW booklet. **Marks**

(a) Find the value of $e^{-\frac{\pi+3}{2}}$, correct to 3 significant figures. **2**

(b) Factorise $50 - 2x^2$. **2**

(c) Simplify $\dfrac{1}{x^2-1} - \dfrac{1}{x-1}$, expressing the answer as a single fraction. **2**

(d) Solve $|3x-5| = 4$. **2**

(e) Find the values of a and b if $\dfrac{1}{3-\sqrt{2}} = a + b\sqrt{2}$. **2**

(f) State the domain and range of $y = \sqrt{2x-1}$. **2**

(g) Find $\displaystyle\int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{3}\, dx$. **3**

Question 12 (15 marks) Start a NEW booklet. **Marks**

(a) Find $\frac{d}{dx}\left(\sqrt{x^3+1}\right)$. **2**

(b) Find the limiting sum of $81-27+9-......$ **2**

(c) Find $\int (3x-2)^5\, dx$. **2**

(d) Find the exact value of $\sin\frac{3\pi}{4}+\cos\frac{7\pi}{6}$. **2**

(e) The gradient function of a curve is given by $3x^2-5$.

If the curve passes through the point $(1,1)$, find its equation. **2**

(f) When a balloon is being filled with helium, its volume at time t is given by

$V=\frac{\pi t^3}{12}\text{ cm}^3$, where t is in seconds.

Find the rate at which the balloon is being filled when $t=1$. **2**

(g) For the parabola $6y=x^2+2x+13$, use completing the square or otherwise to find:

(i) the vertex. **2**

(ii) the focal length. **1**

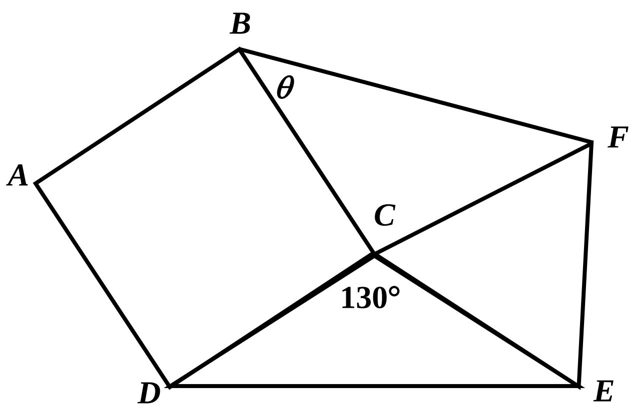
Question 13 (15 marks) Start a NEW booklet. **Marks**

(a) Find $\dfrac{d}{dx}\left(e^{\cos x}\right)$. **1**

(b) Find the shortest distance of the point $(1,-3)$ from the line $2x-5y=4$. **2**

(c) In the figure below, *ABCD* is a square, *CEF* is an equilateral triangle,

$\angle DCE = 130^\circ$ and $DC = CE$.



Find the size of $\angle CBF$, giving clear reasons. **3**

(d) Find the value of k for which the equation $3x^2+10x+k=0$ has:

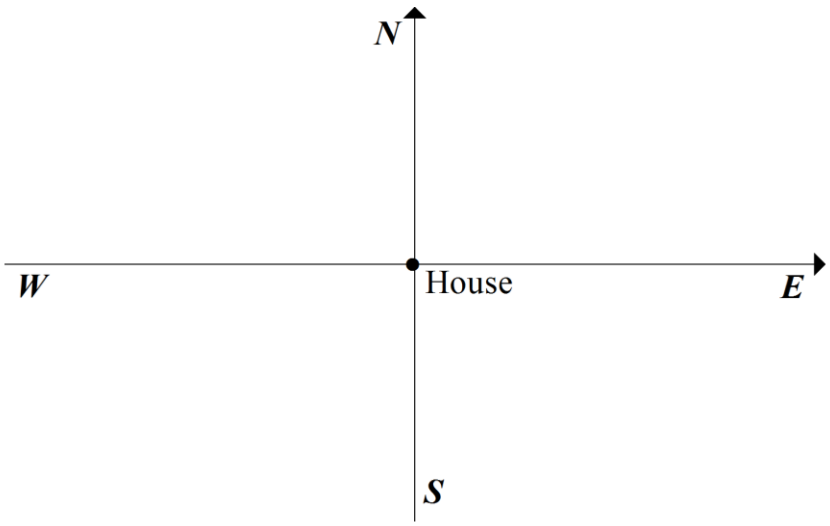
(i) one root which is the reciprocal of the other. **1**

(ii) equal roots. **2**

Question 13 continues on page 9

Question 13 **(**continued) **Marks**

(e)



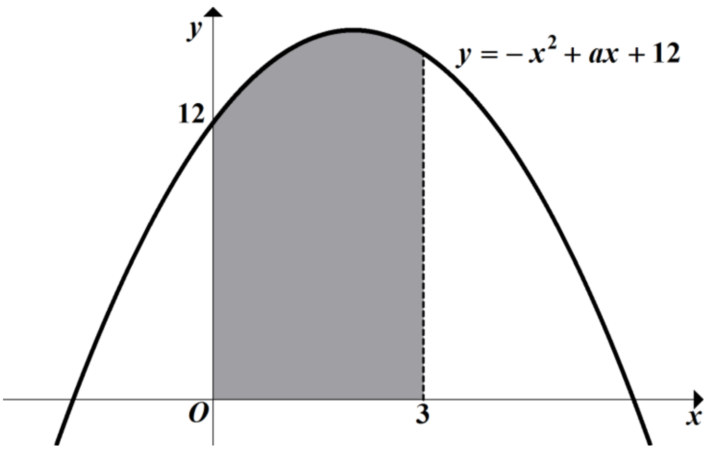
Maeve walks from her house for 6 km, on a bearing of 310° to point *B*.

She then walks on a bearing of 215° to a point *C* which is due west of her house.

(i) Copy the diagram and clearly indicate the information mentioned above. **1**

(ii) Calculate to 1 decimal place the distance of *C* to her house. **2**

(f) Part of the graph of the function $y = -x^2 + ax + 12$, is shown below.



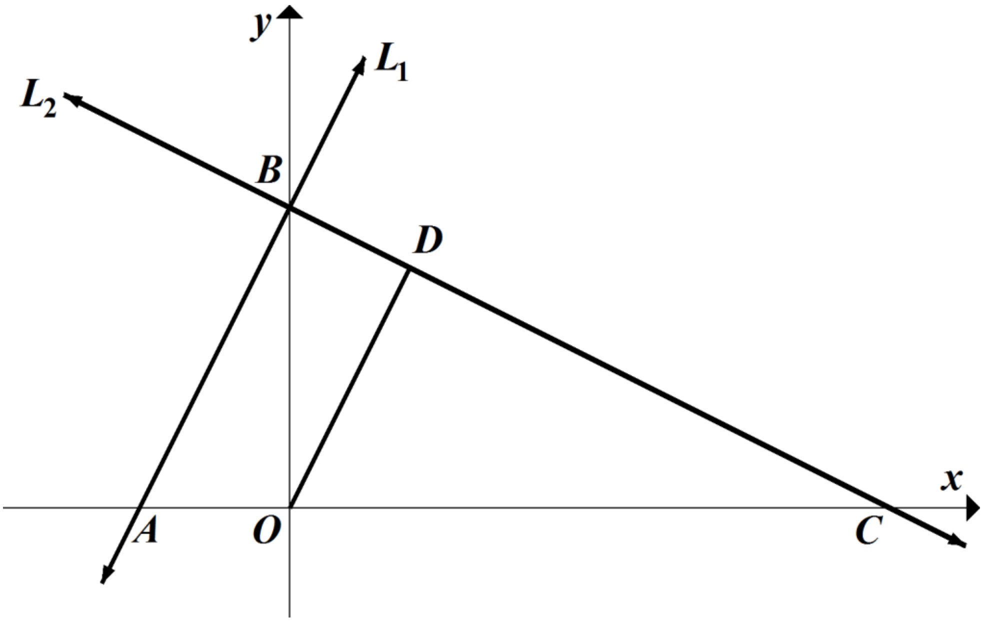
If the shaded area is 45 square units, find the values of *a*. **3**

End of Question 13

Question 14 (15 marks) Start a NEW booklet. **Marks**

(a) In the diagram below, the straight line L_1 has equation $2x - y + 4 = 0$ and cuts the *x*-axis and *y*-axis at *A* and *B* respectively.

The straight line L_2, passing through *B* and perpendicular to L_1 cuts the *x*-axis at *C*. From the origin *O*, a straight line perpendicular to L_2 is drawn to meet L_2 at *D*.



(i) Write down the coordinates of *A* and *B*. **1**

(ii) Show that the equation of L_2 is $x + 2y - 8 = 0$. **2**

(iii) Find the equation of the line *OD*. **1**

(iv) Show that the coordinates of *D* are $\left(\frac{8}{5}, \frac{16}{5}\right)$. **1**

(v) Hence or otherwise, find the area of quadrilateral *OABD*. **2**

Question 14 continues on page 11

Question 14 (continued) **Marks**

(b) (i) State the period for $y = 3\cos\frac{x}{2}$. **1**

(ii) Neatly sketch the graph of $y = 3\cos\frac{x}{2}$ for $0 \le x \le 2\pi$. **2**

(c) Prove that $\frac{\tan\theta}{\sin\theta} - \cos\theta = \tan\theta\sin\theta$. **3**

(d) Michael is attempting to solve the equation $2\ln x = \ln(3x+10)$.

He sets his work out using the following correct steps:

- $\ln x^2 = \ln(3x+10)$
- $x^2 = 3x + 10$
- $x^2 - 3x - 10 = 0$

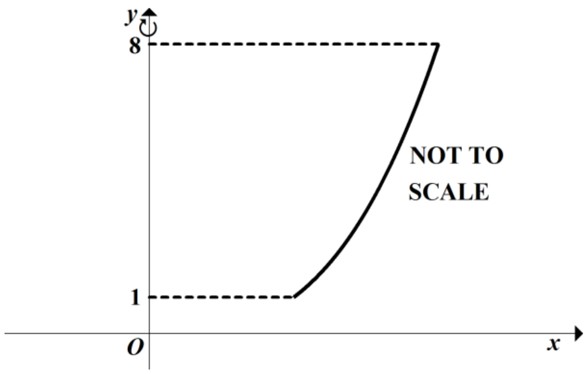
He is now unsure of what to do next.

Complete the solution for Michael. **2**

End of Question 14

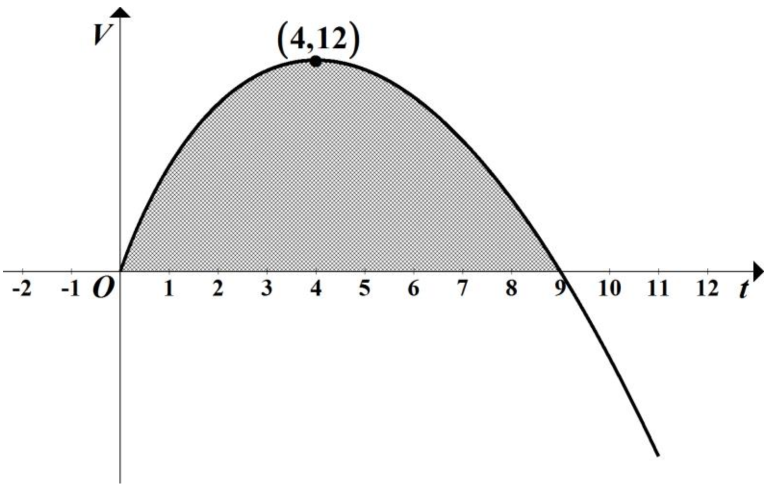
Question 15 (15 marks) Start a NEW booklet. **Marks**

(a) A solid is formed by rotating the curve $y = x^3$ about the y-axis for $1 \le y \le 8$.



Find the volume of the solid in exact form. **3**

(b)



A particle is observed as it moves in a straight line between $t = 0$ and $t = 11$.

Its velocity V m/s at time t is shown on the graph above.

(i) What is the velocity of the particle after 4 seconds? **1**

(ii) What is the particle's acceleration after 4 seconds? **1**

(iii) At what time after $t = 0$ is the particle at rest? **1**

(iv) At what time does the particle change direction? **1**

(v) Explain what the shaded area represents. **1**

Question 15 continues on page 13

Question 15 (continued) **Marks**

(c) In Daniel's first year of fulltime employment his annual salary is $75 000.

At the beginning of the second year his salary increases to $79 000.

His salary continues to increase by $4000 at the beginning of each successive year.

(i) What will Daniel's salary be at the beginning of his 25th year?. **1**

(ii) Calculate Daniel's total earnings after working for 25 years. **1**

(iii) During which year of employment will his total earnings first exceed $2 000 000? **2**

(d) Determine the range of values of x when the curve $y = x^2 \ln x$ is concave up. **3**

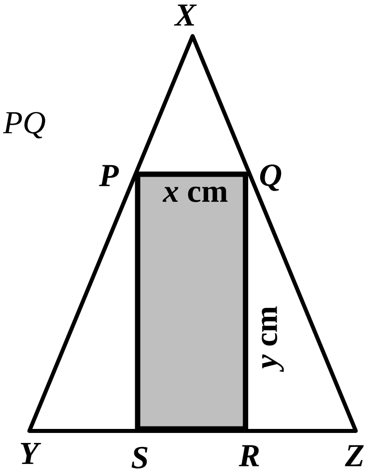
End of Question 15

Question 16 (15 marks) Start a NEW booklet. **Marks**

(a) Evaluate $\int_1^2 \frac{e^x}{e^x - 1}\, dx$ expressing the answer in simplified form. **3**

(b) In ΔXYZ, $XY = XZ = 13$ cm and $YZ = 10$ cm.

A rectangle $PQRS$ is inscribed in the triangle with PQ parallel to YZ. Let $PQ = x$ cm and $QR = y$ cm.



(i) Find the perpendicular height of ΔXYZ. **1**

(ii) Using similar triangles, show that **2**

$$y = 12 - \frac{6x}{5}.$$

(iii) If the area of the rectangle $PQRS$ is A cm², find the maximum value of A. **2**

(c) Consider the function $f(x) = \frac{4}{x^2 + 1}$.

(i) Show that $f(x) = \frac{4}{x^2 + 1}$ is an even function. **1**

(ii) Explain why there are no x-intercepts for $f(x) = \frac{4}{x^2 + 1}$. **1**

(iii) Find any stationary points and determine their nature. **3**

(iv) Neatly sketch the graph of $y = f(x)$. **2**

End of paper

CHS YEAR 12 MATHEMATICS 2U 2016	TRIAL HSC SOLUTIONS

Multiple Choice Section:

1.B 2.A 3.C 4.B 5.D

6.D 7.B 8.C 9.A 10.D

Question 1.

$2c = 1 - a$

$\therefore c = \dfrac{1-a}{2}$ ---------- $\boxed{B}$

Question 2.

$f(1) = 2 - 3 + 4 = 3$

$f(-1) = 2 + 3 + 4 = 9$

$f(1) - f(-1) = -6$ ---------- $\boxed{A}$

Question 3.

decreasing exponentially ---------- $\boxed{C}$

Question 4.

$T_2 = -3 \rightarrow ar = -3$ ---- $\boxed{1}$

$T_5 = 192 \rightarrow ar^4 = 192$ ---- $\boxed{2}$

$\boxed{2} \div \boxed{1} \rightarrow r^3 = -64$

$\therefore r = -4$ ------------ $\boxed{B}$

Question 5.

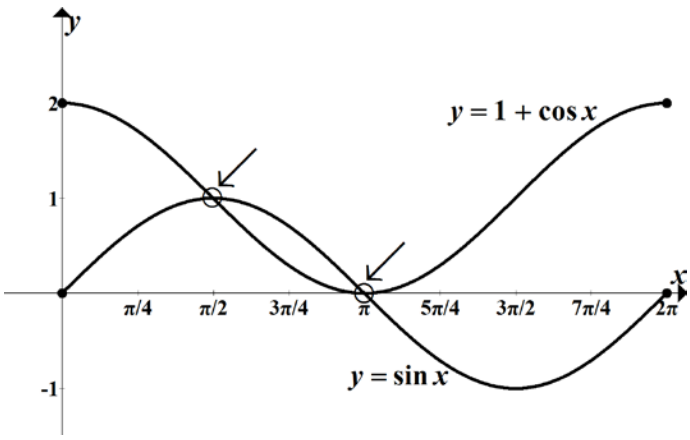
$(x-3)^2 + (y-5)^2 = 4$ ------------ $\boxed{D}$

Question 6.

$A = \dfrac{5-1}{6}\left(f(1) + 4f(3) + f(5)\right)$

$= \dfrac{2}{3}(4a + b)$ ------------ $\boxed{D}$

Question 7.



$\therefore$ 2 points of intersection -------- $\boxed{B}$

Question 8.

Using the test point (3,3) it can be determined that each inequality is true – hence *III* ----- $\boxed{C}$

Question 9.

$OA + OB$ represents the sum of the roots.

$\therefore OA + OB = -\dfrac{b}{a} = -b$ ------------ $\boxed{A}$

Question 10.

$\therefore \log_{10} y - \log_{10} x = \log_{10} z - \log_{10} y$

$\therefore 2\log_{10} y = \log_{10} x + \log_{10} z$

$\therefore \log_{10} y^2 = \log_{10}(xz)$

$\therefore y^2 = xz$ ----------- $\boxed{D}$

Question 11

a) $e^{-\frac{\pi+3}{2}} = 0.04638420$

$\approx 0.0464 \ (3\text{Sig fig})$

b) $50 - 2x^2 = 2\left(25 - x^2\right)$

$= 2(5 - x)(5 + x)$

c) $\frac{1}{x^2-1} - \frac{1}{x-1} = \frac{1}{(x-1)(x+1)} - \frac{x+1}{(x-1)(x+1)}$

$= \frac{1-x-1}{(x-1)(x+1)} = \frac{-x}{(x-1)(x+1)}$

d) $3x-5=4$ or $3x-5=-4$

$3x=9$ or $3x=1$

$\therefore \ x=3$ or $x=\frac{1}{3}$

e) $LHS = \frac{1}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3-\sqrt{2}}$

$= \frac{3+\sqrt{2}}{7} \ \rightarrow \ a=\frac{3}{7}, b=\frac{1}{7}$

f) Domain: all real $x \geq \frac{1}{2}$

Range: all real $y \geq 0$

g) $\int_0^{\frac{\pi}{2}} \sec^2\frac{x}{3} \, dx = 3\left[\tan\frac{x}{3}\right]_0^{\frac{\pi}{2}}$

$= 3\left(\tan\frac{\pi}{6} - \tan 0\right) = \frac{3}{\sqrt{3}}$

Question 12.

a) $\frac{d}{dx}\left(\sqrt{x^3+1}\right) = \frac{1}{2}\left(x^3+1\right)^{-\frac{1}{2}} \times 3x^2$

$= \frac{3x^2}{2\sqrt{x^3+1}}$

b) $81 - 27 + 9 - \ldots\ldots \rightarrow \ a=81, r=-\frac{1}{3}$

$\therefore S_\infty = \frac{81}{1-\left(-\frac{1}{3}\right)} = \frac{243}{4}$

c) $\int (3x-2)^5 \, dx = \frac{(3x-2)^6}{18} + C$

d) $\sin\frac{3\pi}{4} + \cos\frac{7\pi}{6} = \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}$

e) $\frac{dy}{dx} = 3x^2 - 5$

$\therefore y = x^3 - 5x + C$

when $x=1, y=1 \ \rightarrow \ C=5$

$\therefore y = x^3 - 5x + 5$

f) $V = \frac{\pi t^3}{12}\text{cm}^3 \ \rightarrow \ \frac{dV}{dt} = \frac{\pi t^2}{4}$

when $t=1, \frac{dV}{dt} = \frac{\pi}{4}\text{cm}^3/\text{s}$

g) i) $6y = \left(x^2+2x+1\right)+12$

$\therefore \ 6(y-2) = (x+1)^2$

hence the vertex is $(-1, 2)$

ii) for the focal length $4a = 6 \ \rightarrow \ a = \frac{3}{2}$

Question 13

a) $\frac{d}{dx}\left(e^{\cos x}\right) = -\sin x \, e^{\cos x}$

b) $d = \frac{|2\times1 - 5\times-3 - 4|}{\sqrt{2^2+(-5)^2}} = \frac{13}{\sqrt{29}}$

c) $\angle FCE = 60^\circ$ $[\Delta FCE$ is equilateral$]$

$\angle BCD = 90^\circ$ $[\Delta ABCD$ is a square$]$

$\therefore\ \angle BCF = 360^\circ - 130^\circ - 90^\circ - 60^\circ = 80^\circ$

and since $BC = CF$ as $(DC = CE)$

then $\theta = \angle CBF = \dfrac{180^\circ - 80^\circ}{2}$ $[\angle$'s opp = sides$]$

$= 50^\circ$ [in isosceles Δ]

d) i) Product of roots = 1

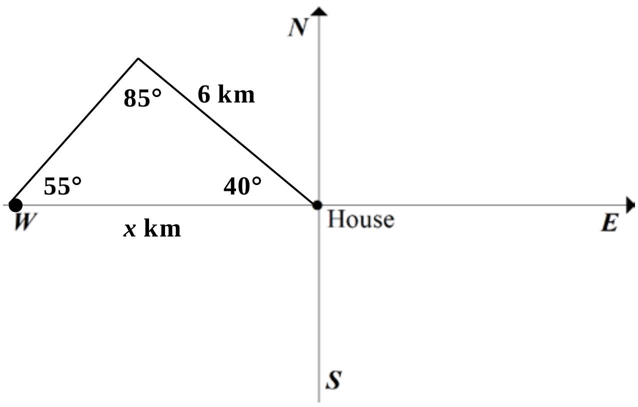
$\therefore\ \dfrac{k}{3} = 1 \rightarrow k = 3$

ii) $\Delta = 0$ for equal roots

$\Delta = b^2 - 4ac = 100 - 12k$

$100 = 12k \rightarrow k = \dfrac{25}{3}$

e)



$\dfrac{x}{\sin 85^\circ} = \dfrac{6}{\sin 55^\circ} \rightarrow x = \dfrac{6\sin 85^\circ}{\sin 55^\circ}$

$\therefore\ x = 7.3\,\text{km}$

f) $\int_0^3 -x^2 + ax + 12\ dx = 45$

$\therefore\ \left[-\dfrac{x^3}{3} + \dfrac{ax^2}{2} + 12x\right]_0^3 = 45$

$-9 + \dfrac{9a}{2} + 36 = 45 \rightarrow a = 4$

Question 14.

a) i) $A(-2,0); B(0,4);$

ii) $m_{L_1} = 2 \rightarrow m_{L_2} = -\dfrac{1}{2}$

$\therefore L_2:\ y - 4 = -\dfrac{1}{2}(x - 0)$ using $B(0, 4)$

$\therefore\ 2y - 8 = -x$

$\therefore\ x + 2y - 8 = 0$

iii) $y = 2x$

iv) Solve $y = 2x$ and $x + 2y - 8 = 0$

$\therefore\ x + 2(2x) - 8 = 0$

$5x = 8 \rightarrow x = \dfrac{8}{5}$

when $x = \dfrac{8}{5}$, $y = 2 \times \dfrac{8}{5} = \dfrac{16}{5}$

v) C has coordinates (8, 0).

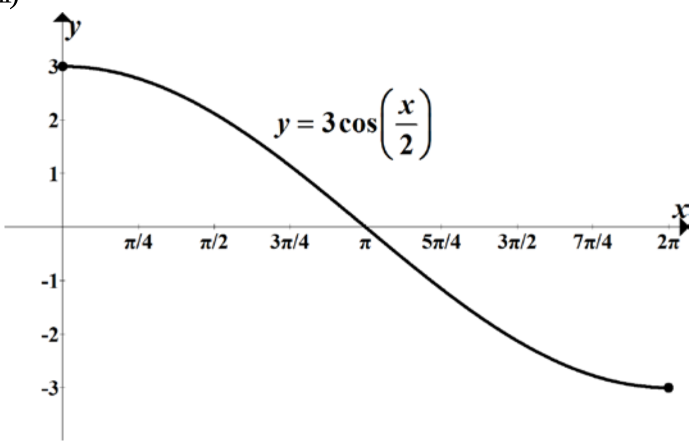
Area of quadrilateral $OABD$ =

Area of $\Delta ABC -$ Area of ΔODC

$= \dfrac{1}{2} \times 10 \times 4 - \dfrac{1}{2} \times 8 \times \dfrac{16}{5} = \dfrac{36}{5} u^2$

b) i) $P = \dfrac{2\pi}{1/2} = 4\pi$

ii)



c) $LHS = \frac{\sin\theta}{\cos\theta} \times \frac{1}{\sin\theta} - \cos\theta$

$= \frac{1}{\cos\theta} - \cos\theta$

$= \frac{1 - \cos^2\theta}{\cos\theta}$

$= \frac{\sin^2\theta}{\cos\theta}$

$= \frac{\sin\theta}{\cos\theta} \times \frac{\sin\theta}{1}$

$= \sin\theta\tan\theta = RHS$

d) $(x-5)(x+2) = 0$

$\therefore\ x = 5$ or $x = -2$

but since $x > 0$ then $x = 5$ only

Question 15.

a) $V = \pi\int_1^8 x^2\,dy$ $\quad \left[y = x^3 \rightarrow x^2 = y^{2/3}\right]$

$\therefore\ V = \pi\int_1^8 y^{\frac{2}{3}}\,dy$

$= \pi\left[\frac{3}{5}y^{5/3}\right]_1^8$

$= \frac{3\pi}{5}\left(8^{5/3} - 1^{5/3}\right)$

$= \frac{3\pi}{5}(31) = \frac{93\pi}{5}\,u^3$

b) i) 12 m/s

ii) 0 m/s^2

iii) $t = 9$s

iv) $t = 9$s

v) The distance travelled in the first 9 seconds.

c) i) $\$75000 + 24 \times 4000 = \$171000$

ii) Total earnings for 25 years =

$\$75000 + \$79000 + \$83000 + ... + \$171000$

$= \frac{25}{2}(\$75000 + \$171000)$

$= \$3075000$

iii) $2000000 = \frac{n}{2}\left(2 \times 75000 + (n-1) \times 4000\right)$

$4000000 = 150000n + 4000n^2 - 4000n$

$\therefore\ 2n^2 + 73n - 2000 = 0$

$n = \frac{-73 \pm \sqrt{73^2 - 4 \times 2 \times -2000}}{4}$

and since $n > 0$, $n \approx 18.26$

hence during the 19$^{\text{th}}$ year.

d) Concave up when $y'' > 0$

$\therefore\ y' = x^2 \times \frac{1}{x} + 2x \times \ln x$

$= x + 2x\ln x$

$\therefore\ y'' = 1 + 2x \times \frac{1}{x} + 2 \times \ln x$

$= 3 + 2\ln x$

$\therefore\ 3 + 2\ln x > 0$

$\therefore\ \ln x > -\frac{3}{2} \quad \rightarrow \quad x > e^{-3/2}$

Question 16.

a) $\int_1^2 \frac{e^x}{e^x - 1}\,dx = \left[\ln\left(e^x - 1\right)\right]_1^2$

$= \ln\left(e^2 - 1\right) - \ln\left(e - 1\right)$

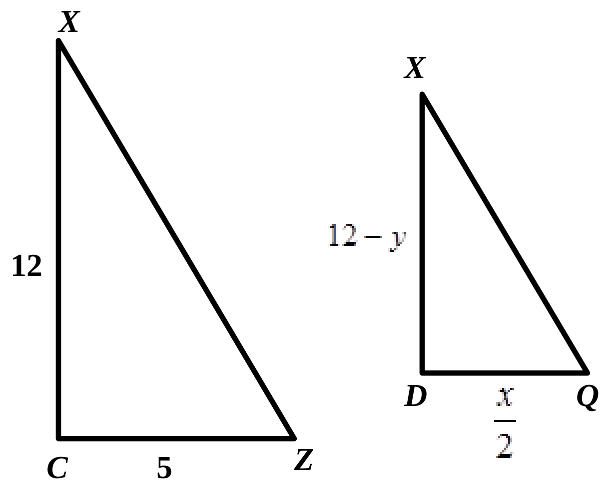
$= \ln\frac{(e-1)(e+1)}{(e-1)}$

$= \ln(e+1)$

b) i) Perpendicular height is 12 cm (Pythagoras)

ii) Let D be the midpoint of PQ.

$\therefore\ DQ = \frac{x}{2}$ and $XD = 12 - y$.



The triangles are similar (equiangular), hence corresponding sides are in the same ratio.

$\therefore\ \frac{12 - y}{x/2} = \frac{12}{5}$

$\therefore\ 6x = 60 - 5y$

$\therefore\ y = 12 - \frac{6x}{5}$

iii) $A = xy \quad \rightarrow \quad A = x\left(12 - \frac{6x}{5}\right)$

$\therefore\ A = 12x - \frac{6x^2}{5}$

For a maximum area $\frac{dA}{dx} = 0$ and $\frac{d^2A}{dx^2} < 0$.

$\frac{dA}{dx} = 12 - \frac{12x}{5};\ \frac{d^2A}{dx^2} = -\frac{12}{5} < 0$ hence maximum.

$\therefore\ 12 - \frac{12x}{5} = 0 \quad \rightarrow \quad x = 5\text{cm}$

Hence maximum area $= 5 \times \left(12 - \frac{6 \times 5}{5}\right)$

$= 30\text{cm}^2$

c) i) A function is even if $f(x) = f(-x)$.

$f(-x) = \frac{4}{(-x)^2 + 1}$

$= \frac{4}{x^2 + 1} = f(x)$, hence even.

ii) x-intercepts occur when $f(x) = 0$, and since the numerator is constant then $f(x) \neq 0$.

iii) $f(x) = 4\left(x^2 + 1\right)^{-1}$

$\therefore\ f'(x) = -8x\left(x^2 + 1\right)^{-2} = \frac{-8x}{\left(x^2 + 1\right)^2}$

For stationary points $f'(x) = 0 \quad \rightarrow \quad x = 0$.
When $x = 0$, $y = 4$.
Test for maximum or minimum:

x	-1	0	1
$f'(x)$	2	0	-2
	/	—	\

Hence $(0, 4)$ is a maximum stationary point.

iv)